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Coordinated Lot-sizing and Dynamic Pricing under a Supplier All-units Quantity Discount

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Abstract

We consider an economic order quantity model where the supplier offers an all-units quantity discount and a price sensitive customer demand. We compare a decentralized decision framework where selling price and replenishment policy are determined independently to simultaneous decision making. Constant and dynamic pricing are distinguished. We derive structural properties and develop algorithms that determine the optimal pricing and replenishment policy and show how quantity discounts not only influence the purchasing strategy but also the pricing policy. A sensitivity analysis indicates the impact of the fixed-holding cost ratio, the discount policy, and the customers' price sensitivity on the optimal decisions.

Keywords: Dynamic Pricing, Economic Order Quantity, Quantity-Discount, Procurement-Inventory Policies, Marketing-Operations Interface

1 Introduction

In recent years, the coordination of marketing and operations decisions has received a lot of attention. However, in practice these two decision areas are usually still independent. Even when the performance of marketing and operations is independently optimized in order to achieve their respective best operating level, it may lead to sub-optimal performance for the firm as a whole (Ghose and Mukhopadhyay, 1993). In particular, the coordination of pricing strategies and operations decisions is still at an early stage and offers significant opportunities for improving supply chain performance. In this context, *dynamic pricing strategies* use intertemporal price discrimination to achieve a better match of supply with demand. On the supply side, the order quantity is a function of customer demand and fixed and variable purchasing costs. On the sales side, demand depends on the charged selling price. Therefore, determining the optimal dynamic selling prices and optimal order quantities is interdependent. It is common in practice that marketing determines the selling price first without taking into consideration overhead

purchasing or a supplier quantity discount. Then, given the selling price and the resulting demand forecast, operations decides on order quantities. However, due to the fact that marketing influences demand by the charged price and operations is responsible for satisfying this demand, price and procurement decisions are strongly connected and should be decided simultaneously. Fleischmann *et al.* (2004) review the linkage between pricing and operational decisions. They indicate different drivers for dynamic pricing strategies like promotions and other marketing-related instruments and more operations-driven pricing strategies known from supply chain coordination.

It is a common practice in that suppliers offer a discount for large order quantities. For this kind of price discrimination, a supplier designs a menu of price-quantity pairs and customers select their optimal purchasing volume. Dolan (1987) and Wilcox *et al.* (1987) provide several reasons for a firm to offer quantity discounts from both a marketing and an operations management point of view. If the supplier faces high setup costs which lead to a large lot-size and high holding costs, quantity discounts may reduce the inventory level immediately after stocking due to larger customer orders. Furthermore, suppliers offer quantity discounts

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for a better utilization of idle capacity in order to achieve economies of scale in manufacturing. From a marketing perspective, quantity discounts are used to stimulate sales, e.g., *Neslin et al.* (1995). From a financial point of view, the time value of money is taken into consideration. Because of the offered quantity discount, buyers decide to buy earlier and a larger quantity. Therefore, revenues are available earlier for possible reinvestment (*Beraneck, 1967*).

In this paper we consider an economic order quantity (EOQ) model where the supplier offers an all-units quantity discount (AQD). This model is applicable to all goods that are purchased from a supplier that offers a quantity discount, e.g., durable consumer or industrial goods. We compare a decentralized decision framework where first marketing determines the optimal pricing strategy and then operations optimizes the replenishment policy to coordinated decision making where the retailer decides on pricing strategy and replenishment policy simultaneously. Hereby, we distinguish between two pricing strategies. In cases of constant pricing, the retailer determines the optimal selling price that is constant over an infinite planning horizon. In case of dynamic pricing the retailer varies the selling price over time. If price changes are free of charge, the optimal strategy is to change the price continuously over time (*Rajan et al., 1992*). If price changes are associated with costs, a continuous price adjustment is not beneficial. Therefore, there exists a finite optimal number of price changes. *Netessine* (2006) provides a variety of practical situations which cause that companies change prices only a limited number of times throughout the sales period. We analyze the benefits from coordinated dynamic pricing and replenishment compared to coordinated constant pricing and replenishment and to decentralized decision making. We assume that the number of price changes over an order cycle is given. However, in order to determine the optimal number of price changes, the results of *Transchel and Minner* (2005) can be used. They analyze a joint dynamic pricing and EOQ lot-sizing model without consideration of a supplier quantity discount and show that the optimal number of price changes and the optimal pricing and replenishment policy can be optimized by a two-stage program. That is, at the second stage, for a given number of price changes, the retailer determines the optimal pur-

chasing and pricing strategy. At the first stage, the number of price changes is optimized anticipating the optimal purchasing and pricing strategy of the second stage. *Transchel and Minner* (2005) show how dynamic pricing can enhance operational efficiency by increasing the demand rate when inventories are high. The benefits of exploiting supply quantity discounts significantly depend on whether variable purchasing price reductions can offset additional holding costs from ordering minimum required quantities. Therefore it appears promising to use dynamic pricing here, too, to reduce the impact of holding costs. Furthermore, we extend *Eliashberg and Steinberg* (1993) who compare sequential and simultaneous optimization of lot-size and (constant) selling price without quantity discounts. They show that the optimal selling price in the case of simultaneous optimization is larger than in the case of sequential optimization which, in turn, results in a lower order frequency. This property does not necessarily hold if the supplier offers a quantity discount. We develop optimization models for three different decision frameworks: the *decentralized framework* where marketing and operations optimize independently, the *coordinated-constant framework* where the retailer optimizes a constant price and the order quantity simultaneously, and the *coordinated-dynamic framework* where the retailer employs a finite number of price changes over an order cycle. We provide analytical properties of the objective functions and present algorithms for determining the optimal replenishment policy and price strategy for the *coordinated-constant* and the *coordinated-dynamic framework*, respectively.

The remainder of the paper is organized as follows. After a literature review in Section 2, we introduce the models and the underlying assumptions and properties in Section 3. Section 4 compares the three frameworks for the special case of a linear price-response function. In a numerical example presented in Section 5 we show the benefit of coordinated decision making and in particular dynamic pricing compared to a decentralized decision framework. Furthermore, we show the impact of changing fixed costs, discount policy, and price sensitivity on the optimal decisions and the resulting benefit of coordination. Finally, we conclude the results, discuss limitation of the model, and indicate directions for further research in Section 6.

2 Literature review

Several contributions analyze joint pricing and manufacturing situations. One of the earliest papers of integrated marketing/pricing and manufacturing decisions is by *Whitin* (1955) who shows that simultaneous decision-making on a constant price and ordering decision leads to significant profit increases compared to decentralized decisions. *Eliashberg and Steinberg* (1993) provide a comprehensive review of problems at the interface between marketing and operations. They compare differences between a decentralized and a coordinated optimization problem. In their overview, *Bitran and Caldentey* (2003) summarize dynamic pricing policies and their relation to revenue management. *Elmaghraby and Keskinocak* (2003) provide a review of literature and current practices in dynamic (intertemporal) pricing. *Chan et al.* (2004) review and classify papers according to a number of characteristics, e.g., stochastic vs. deterministic parameters and single vs. multiple products.

Benton and Park (1996) and *Munson and Rosenblatt* (1998) provide literature reviews on quantity discounts where they classify the literature under several discount schemes, different perspectives (buyer, supplier, joint), and other criteria like planning horizon and number of products. The impact of quantity discounts on the economic ordering decision has been investigated in several settings. In the literature, the most common discount policies are the *all-units* and the *incremental-units* discount policies. When the supplier offers an all-units quantity discount (AQD), the reduced purchasing price applies to the entire order quantity once the order quantity reaches a critical breakpoint. An incremental-units discount, however, only applies to all units in excess of a particular breakpoint. *Hadley and Whitin* (1963) develop a procedure for determining the optimal economic order quantity for both all-units and incremental units discount schemes. This approach is included in almost every textbook on operations management and assumes that the demand rate is known and constant over an infinite planning horizon and that the decision maker follows the objective to minimize average costs. *Gupta* (1988) provides an improved procedure for determining the optimal lot-size by consid-

ering an upper bound for the relevant cost and *Goyal and Gupta* (1990) propose a further simplification which requires only a few EOQ calculations for determining the optimal lot-size. In *Abad* (1988a) and *Abad* (1988b) the simultaneous optimization of lot-size and selling price when the supplier offers an all-units or an incremental units discount is analyzed. He develops a procedure for determining the optimal lot-size and the optimal selling price. *Burewell et al.* (1991) extend the model of *Abad* (1988a) and allow for planned inventory shortages. They derive a similar procedure as *Abad* (1988a) to determine the optimal lot-size and selling price for two classes of demand functions, iso-elastic and linear. More recently, discount pricing schedules have received growing attention to improve the coordination between vendors and buyers, see e.g., *Weng* (1995), *Rubin and Benton* (2003), and *Wang* (2005).

Inventory problems with simultaneous optimization of the replenishment policy and a dynamic pricing strategy where only a limited number of price changes is allowed has been investigated less frequently. The main motivation for considering a limited number of price changes are organizational costs associated with each price change. *Abad* (1997) formulates a model where the reseller responds to a temporary price reduction of the supplier by an adjustment of the retailer's own selling price. *Abad* (1997) considers that the reseller is allowed to charge two selling prices in each order cycle. The presented model optimizes the first (discounted) selling price and fixes the second selling price to the optimal constant price. He presents a procedure where a temporarily reduced selling price yields a higher cycle profit than in the case of a static selling price. As shown in *Transchel and Minner* (2005), the majority of benefits from dynamic pricing can be captured by very few different price levels and, therefore, a discrete number of changes balances benefits and costs of price changes. *Netessine* (2006) analyzes a pricing problem with a limited number of price changes in a dynamic, deterministic environment and a capacity constraint. He characterizes the impact of capacity constraints on the optimal prices and the timing of price changes and provides several comparative statics results.

3 Model

3.1 Assumptions

Consider a monopolistic retailer who is selling a product on a single market over an infinite planning horizon. We assume that customer demand follows a function of the selling price P and arrives continuously at a rate of $D(P)$ which is a differentiable and strictly non-increasing function in P with $D(P) \geq 0$ and $D'(P) < 0$. Furthermore, let ε define the price elasticity of $D(P)$ with $\varepsilon = -\frac{D'(P)}{D(P)}P$ which is the percentage change in demand in response to a percentage change in price. Based on the price elasticity, we define a class of price-response functions as follows:

Definition 1 A price-response function $D(P)$ has an increasing price elasticity (IPE), if $\frac{\partial \varepsilon}{\partial P} \geq 0$.

The intuition behind the IPE property is that with a price increase by a certain percentage demand decreases by a larger percentage. At every point in time, the demand rate depends only on the current price, i.e., customers are willing to buy as soon as the price is below their individual reservation price. Let \bar{P} denote the market reservation price with $\bar{P} \leq \infty$. We assume that by charging a reduced selling price at the beginning of an order cycle customers do not behave strategically and buy in expectation of future prices (forward buying and postponement).

We follow the assumptions of the EOQ model. With the release of any single order there is an associated setup cost F . Furthermore, the supplier has no capacity constraints and the overall order quantity is delivered in one shipment without any delay. Products delivered but not yet sold are kept in inventory subject to holding cost per unit and unit of time. The replenishment orders are placed in batches of size Q every T periods over an infinite planning horizon. Backorders are not permitted. The supplier offers a regular purchasing price c_0 per unit and an AQD schedule with $l = 0, \dots, L$ different purchasing price categories where the discount is r_l percent per unit if the order quantity is larger than or equal to a breakpoint quantity \bar{Q}_l . The AQD policy with multiple breakpoints is characterized by a vector

$$\{(r_0, \bar{Q}_0), (r_1, \bar{Q}_1), \dots, (r_L, \bar{Q}_L) \\ | r_0 < r_1 < \dots < r_L, \bar{Q}_0 < \bar{Q}_1 < \dots < \bar{Q}_L\}$$

with $r_0 = \bar{Q}_0 = 0$. Let $c_l := (1 - r_l)c_0$ be the reduced purchase price for a unit if the order quantity $Q \in [\bar{Q}_l, \bar{Q}_{l+1})$ with $c_0 > c_1 > \dots > c_L$ and $\bar{Q}_{L+1} = \infty$. Inventory holding costs depend, among others, on the cost of capital that, in turn, depends on the purchase price c_l and are denoted by h_l per unit and unit of time. We assume that h_l is an increasing function of c_l .

3.2 Decentralized decisions

Assume that the selling price and the purchasing strategy are determined by separated decision-making units. First, marketing optimizes the selling price and generates customer demand. Then, given this demand forecast, operations determines the optimal replenishment policy taking into account the supplier's quantity discount. Marketing does not take into account fixed ordering costs and does not anticipate purchasing price discounts as a result of order quantities because the discount that is actually applied is unknown until the operations decision is taken. Marketing's objective function is as follows:

$$(1) \quad \tilde{\Pi}(P) = (P - c_0)D(P).$$

The optimal selling price \tilde{P}^* is obtained from the first-order condition $P + \frac{D(P)}{D'(P)} = c_0$. Given the demand forecast $\tilde{D}^* = D(\tilde{P}^*)$, operations minimizes average costs of replenishment and inventory taking into account the supplier quantity discount. *Hadley and Whitin* (1963) developed a two-stage algorithm in order to determine the optimal replenishment policy. A first stage iteratively calculates the constrained economic order quantity \tilde{Q}_l^* and the resulting costs starting from the highest discount c_L until the first index l_0 is found where the solution satisfies $\tilde{Q}_{l_0}^* \geq \bar{Q}_{l_0}$ and $\tilde{Q}_l^* < \bar{Q}_l$ for all $l > l_0$. Thus,

$$(2) \quad \tilde{Q}_{l_0}^* = \sqrt{\frac{2FD(\tilde{P}^*)}{h_{l_0}}} \geq \bar{Q}_{l_0} \quad \text{and} \\ \tilde{C}_{l_0}^* = \sqrt{2Fh_{l_0}D(\tilde{P}^*)}.$$

At a second stage, the cost of this inner solution $\tilde{C}_{l_0}^*$ is compared with the costs of all breakpoint quantities larger than $\tilde{Q}_{l_0}^*$, i.e., $\tilde{C}_l(\bar{Q}_l)$ for $l = l_0 + 1, \dots, L$ with

$$\tilde{C}_l(\bar{Q}_l) = c_l D(\tilde{P}^*) + \frac{h_l}{2} \bar{Q}_l + F \frac{D(\tilde{P}^*)}{\bar{Q}_l}.$$

Thus, the optimal profit in the case of decentralized decision making is as follows:

$$\tilde{\Pi}^* = (\tilde{P}^* - c_0)D(\tilde{P}^*) - \min\{\tilde{C}_{l_0}^*, \tilde{C}_l(\bar{Q}_l) \mid l = l_{0+1}, \dots, L\}.$$

3.3 Coordinated-constant decision

In case of coordinated decision making, we simultaneously optimize lot-size Q and selling price P . The optimization problem for a particular purchasing price c_l is given by

$$(3) \quad \Pi_l^*(P, Q) = \max_{P, Q} \left((P - c_l)D(P) - \frac{h_l}{2}Q - F\frac{D(P)}{Q} \right)$$

$$(4) \quad \text{s.t. } Q \geq \bar{Q}_l.$$

A relaxation of (4) and differentiating (3) with respect to Q and P yields the necessary first-order conditions for an inner solution (e.g., see *Whitin* (1955) and *Eliashberg and Steinberg* (1993))

$$(5) \quad Q_l^* = \sqrt{\frac{2FD(P_l^*)}{h_l}} \quad \text{and} \quad P_l^* + \frac{D(P_l^*)}{D'(P_l^*)} = c_l + \frac{F}{Q_l^*}$$

where the optimal selling price P_l^* , conditional that the purchasing price is c_l , is represented as an implicit function of the optimal order quantity. For a particular purchasing price c_l the optimal order quantity is only feasible if $\bar{Q}_l \leq Q_l^*$. Given this constraint, a transformation of Q_l^* from (5) yields that an inner solution P_l^* is only feasible if P_l^* is lower than a break price \bar{P}_l for a given unit purchasing cost c_l :

$$(6) \quad P_l^* \leq D^{-1} \left(\frac{h_l \bar{Q}_l^2}{2F} \right) =: \bar{P}_l$$

where $D^{-1}(\cdot)$ denotes the inverse of the price-response function. The inverse function does exist due to the fact that $D(P)$ is a strictly monotone function. Using this result, the AQD policy can be characterized by

$$\{(r_0, \bar{P}_0), (r_1, \bar{P}_1), \dots, (r_L, \bar{P}_L) \mid r_0 < r_1 < \dots < r_L, \bar{P}_0 > \bar{P}_1 > \dots > \bar{P}_L\}$$

where $\bar{P}_0 = \bar{P}$. Substituting the optimal order quantity into (3), the two-variable problem is reduced to a single-variable problem that only depends on P :

$$(7) \quad \Pi_l(P) = (P - c_l)D(P) - \sqrt{2Fh_l D(P)},$$

$$(8) \quad \text{s.t. } P \leq \bar{P}_l.$$

Properties of an optimal pricing and lot-sizing policy

The following properties characterize the profit function (7).

Property 1 For $P \geq 0$, (7) is either a concave-concave function or a strictly concave function of P with $\lim_{P \rightarrow 0} \Pi_l(P) < 0$ and $\lim_{P \rightarrow \bar{P}} \Pi_l(P) = 0$.

The proof of Property 1 is given in Appendix A.1. *Abad* (1988a) reduces (3) to a single-variable problem that only depends on the order quantity Q . He shows that if the first-order condition with respect to P yields a closed-form solution $P^*(Q)$, as it is the case for linear and iso-elastic price-response functions, the reduced profit function is a convex-concave function of Q and develops a procedure to determine the optimal price and lot-size. Based on this result, *Abad* (1988a) gives an algorithm to determine the overall optimal lot-size. Note that we do not need the required condition of having a closed-form solution for $P^*(Q)$.

Property 2 For an arbitrary fixed selling price P , (7) is a strictly decreasing function of c_l . Therefore, the profit functions Π_l and $\Pi_{l'}$ for different unit purchasing costs c_l and $c_{l'}$ do not intersect and $\Pi_l(P) > \Pi_{l'}(P)$ for all P and $c_l < c_{l'}$.

From the partial derivative of (7) with respect to c_l and the assumption that h_l increases in c_l , it is easy to verify that $\frac{\partial \Pi_l}{\partial c_l} < 0$. Implicitly, it follows that the profit function $\Pi_l(P) > \Pi_{l'}(P)$ for $c_l < c_{l'}$.

Theorem 1 Let l_0 be the largest index of a discount where the local optimum $Q_{l_0}^*$ is feasible, i.e., $\bar{Q}_{l_0} \leq Q_{l_0}^*$ and $\bar{Q}_l > Q_l^*$ for all $l = l_0 + 1, \dots, L$. Then for all $l < l_0$, $\Pi_l(P) < \Pi_{l_0}(P)$ for all P .

The proof follows directly from Properties 1 and 2 with the implication that all discounts that are lower than the discount r_{l_0} can be omitted for the determination of the optimal solution. If $\Pi_l(P_l^*) < 0$ for all c_l , then $Q^* = 0$. To find the optimal value of P , we have to find the profit maximizing price in each interval $(\bar{P}_{l+1}, \bar{P}_l]$ and compare these profits to determine the global optimum.

For the following illustration we assume that the supplier offers a single price break \bar{Q}_1 . Then, there are 3 cases for the optimal price P^* .

1. The free local optimum P_1^* for the reduced purchasing price c_1 is a feasible solution, i.e., $P_1^* \leq \bar{P}_1$.

